ECS 20 – Fall 2021 – P. Rogaway Pigeonhole Principle

Basic statement

Pigeonhole Principle (v1): If N pigeons roost in n holes, with $n < N$, then some two pigeons roost in the same hole.

Pigeonhole Principle (v2): If $f : A \to B$ where A and B are finite sets and $|A| > |B|$, then f is **not** injective: there are points $a, a' \in A$ such that $a \neq a'$ and $f(a) = f(a')$.

Ex 1. Any room with 3 or more people has some two of the same gender. Assume for this statement that each person identifies as either "male" or "female" but not both.

Ex 2. 20 people at a party, some two have the same number of friends.

Proof idea: two cases: no person knows everyone; some person knows everyone. Then there will be 0..18 possible # of friends in the first case, and 1..19 number of friends in the second case. Apply PHP in each case.

Pigeonhole Principle (strong form): If $f : A \to B$ where A and B are finite sets, then some point $y \in B$ must have at least $\lceil |B| / |A| \rceil$ preimages under f.

Eg 3: if 100 pigeons roost in 30 holes, some hole has at least 4 pigeons roosting therein.

More examples:

Ex 4: Given five points inside the square whose side is of length 2 feet, prove that two are less than 1.5 feet apart.

Proof idea: divide square into four 1 x 1 cells. The diameter of each cell is $2^{0.5}$ which is less than 1.5.

Ex 5. Prove that for any five points on a sphere, some four must lie on the same hemisphere. Assume that the boundary of the hemisphere is on both hemispheres.

Proof: choose any two of them and draw the great circle route that connects them (take a plane cutting through those two points and the center of the sphere, and see where it intersects the sphere). Three points remain. Two must be on one side of the sphere; one will be on the other. The two points on one side of the sphere, together with the two equatorial points on the great circle, are four points within the same hemisphere.

Ex 6: In any list of 10 integers $a_1, ..., a_{10}$ there's a subsequence of consecutive numbers whose sum is divisible by 10.

Consider the ten sums

$s_1 = a_1$

$s_2 = a_1 + a_2$

...

$s_{10} = a_1 + a_2 + \ldots + a_{10}$

numbers in the list. If any of these divisible by 10, then we are done. Otherwise, each is congruent to a number between 1 and 9 mod 10. So two of these values are congruent to the same number (mod 10): $a_i = a_j$ (mod 10) with $i < j$. Eg, maybe

$s_3 = a_1 + a_2 + a_3$

and

$s_5 = a_1 + a_2 + a_3 + a_4 + a_5$

are both congruent to 7 (mod 10). But then $s_5 - s_3 = 0$ (mod 10), which would mean that $s_4 = 0$ (mod 10). Generalizing,

$a_{i+1} + \ldots + a_j = 0 \pmod{10}$

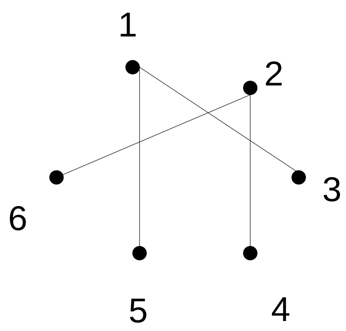
Ex 7: Devon picks 7 different numbers from {1, 2, 3, . . . , 10, 11}. Prove some pair adds up to 12.

Proof. Consider the following partition of Devon's available numbers:

{1, 11}, {2, 10}, {3, 9}, {4, 8}, {5, 7}, {6}.

Since there are only 6 sets in this partition, two of Devon's 7 numbers must be in the same set, hence that set is not {6}. But all the sets with two numbers have elements summing to 12.

Ex 8. (repeated from beginning of term) In any room of 6 people, there are 3 mutual friends or 3 mutual strangers (Ramsey theorem, $R(3,3)=6$)



Consider person #1. Five people are either friends or non-friends with person #1. At least one of those sets has 3 people.

- 3 people are non-friends with 1. If two of them don't know one another, we are done. If all three know one another, we are done.
- 3 people are friends with 1. If two of them know one another, we are done. If all three don't know one another, we are done.

$R(4,4) = 18$ (1955)
$R(5,5)$ open!